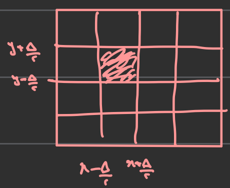
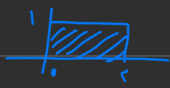
انواع جفت متغیر تصادفی : (X,Y)

(X و Y هردو پیوسته)



$$f_{XY}(x,y) = \lim_{\Delta\to 0} \frac{P[X,Y \in (\text{مربع})]}{\text{مساحت مربع }(\Delta^2)}$$

مثال : فرض کنید یک نقطه به شکل یکنواخت از مستطیل



انتخاب شده . $? = f_{XY}(x,y)$

راه ۱ :

$$f_{XY}(x,y) = \lim_{\Delta\to 0} \frac{\Delta^2}{2\Delta^2} = \boxed{\frac{1}{2}}$$

$$\iint f_{XY}(x,y)\,dx\,dy = 1 \longleftrightarrow \sum_{x,y} P_{XY}(x,y) = 1$$

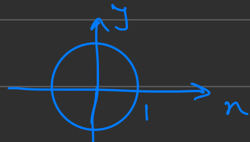
★ منظور از احتمال یکنواخت در حالت پیوسته :
چگالی احتمال در ناحیه مورد نظر یکسان است .

راه ۲

$$\int_0^2\int_0^1 c\,dx\,dy = 2c = 1 \longrightarrow c = \frac{1}{2}$$

$$c = f_{XY}(x,y) = \boxed{\frac{1}{2}}$$

مثال ۲ . یک نقطه یکنواخت از دایره



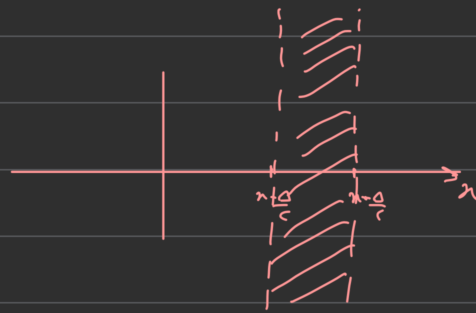
$$1 = \iint c\,dx\,dy = \pi c$$

$$\longrightarrow c = f_{XY}(x,y) = \boxed{\frac{1}{\pi}}$$

$f_X(x) = ?$

$$\Delta f_X(x) = P\left[x - \frac{\Delta}{2} \leq X \leq x + \frac{\Delta}{2}\right]$$

$$= \int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} \int_{-\infty}^{\infty} f(x,y)\, dy\, dx$$



$$P(X = x) = \sum_y P(X = x; Y = y) \longleftrightarrow f_X(x) = \int f_{XY}(x;y)\, dy$$

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \underbrace{f(x,y)}_{\downarrow\ \frac{1}{\pi}}\, dy = \frac{2\sqrt{1-x^2}}{\pi}$$

ادامه مثال قبل.

$f_{Y|X}(y|x) = ?$

$$f_{Y|X}(y|x) \simeq \frac{P\left[y - \frac{\Delta}{2} \leq Y \leq y + \frac{\Delta}{2} \mid X = x\right]}{\Delta}$$

$$= \frac{P\left[Y \in y \pm \frac{\Delta}{2} ; X = x\right]}{P[X = x]\ \Delta} = \frac{P\left[Y \in y \pm \frac{\Delta}{2} ; X \in x \pm \frac{\Delta}{2}\right]}{P\left[X \in x \pm \frac{\Delta}{2}\right]\ \Delta}$$

$$= \frac{f(x,y)\, \Delta^2}{f(x)\, \Delta^2} = \frac{f(x,y)}{f(x)}$$

ادامه مثال قبل:

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f(x)} = \frac{\frac{1}{\pi}}{\frac{2\sqrt{1-x^2}}{\pi}} = \frac{1}{2\sqrt{1-x^2}}$$

$$\sum_y P_{Y|X}(y|x) = 1 \longleftrightarrow \int f_{Y|X}(y|x)\,dy$$

$$P_{XY}(x,y) = P[X=x; Y=y]$$

$$\sum_{x,y} P_{XY}(x,y) = 1$$

$$P[(X,Y)\in S] = \sum_{(x,y)\in S} P_{XY}(x,y)$$

$$P(X=x) = \sum_y P(X=x; Y=y)$$

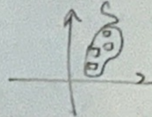
$$P_{Y|X}(y|x) = \frac{P_{XY}(x,y)}{P_X(x)}$$

$$\sum_y P_{Y|X}(y|x) = 1$$

$$\delta\, f_{XY}(x,y) \approx P[(X,Y) \in \delta]$$

$$\iint f(x,y)\,dx\,dy = 1$$

$$P[(X,Y)\in S]$$



$$= \iint_{(x,y)\in S} f_{XY}(x,y)\,dx\,dy$$

$$f_X(x) = \int f_{XY}(x,y)\,dy$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$\int f_{Y|X}(y|x)\,dy = 1$$

$$E[\varphi(x,y)] = \iint \varphi(x,y)\, f(x,y)\, dx\, dy$$

$$\varphi(x,y) = X + Y$$

$$E[X+Y] = E[X] + E[Y]$$

$$E[\varphi(x,y) \mid X=x] \xrightarrow{\text{گسسته}} = \sum_{y} P(y \mid x)\, \varphi(x,y)$$

$$\xrightarrow{\text{پیوسته}} = \int f_{Y|X}(y \mid x)\, \varphi(x,y)\, dy$$

$$E[\varphi(X,Y)] = E\Big[E[\varphi(X,Y) \mid X]\Big] = E\Big[\underbrace{\int_{-\infty}^{\infty} f(y \mid x)\, \varphi(x,y)\, dy}_{h(x)}\Big]$$

$$= \int h(x)\, f_X(x)\, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x,y)\, f(y \mid x)\, dy$$

$$= \iint \varphi(x,y)\, f(x,y)\, dy\, dx$$

استقلال : گسسته :

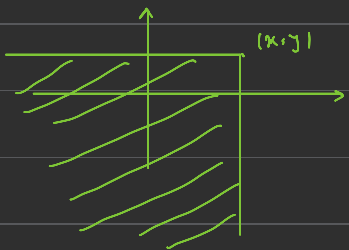
$$P[X=x; Y=y] = P[X=x]\,P[Y=y]$$

پیوسته :

$$f_{XY}(x,y) = f_X(x)\,f_Y(y)$$

محاسبه تابع چگالی احتمال مشترک :

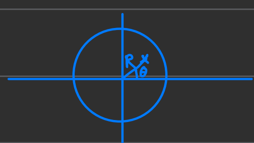
$$F_{XY}(x,y) = P[X \le x;\ Y \le y]$$



$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

مثال :

نقطه‌ای یکنواخت از دایره واحد :



$$f_R(r) = 2r$$

$$f_\Theta(\theta) = \frac{1}{2\pi}$$

$$F_{R\Theta}(r,\theta) = P[R \le r;\ \Theta \le \theta] = \frac{r^2\theta}{2\pi}$$

$$f_{R\Theta}(r,\theta) = \frac{r}{\pi}$$